**Motion in two Dimensions**

**Motion in two Dimensions:** When the lines of motion are all parallel to fixed plane (say ) and velocity at the corresponding points of all planes to  plane has the same magnitude and direction, the motion is said to be two dimensional. The fact is shown graphically in Fig-1.

























Fig.-1: Two dimensional motion.

**Question-01:** Define stream function or current function.

OR

Show that the stream function is constant along the stream lines.

**Answer: Stream function or Current function:** In the case of two dimensional motion, the velocity vector  is a function of , ,  but not of . Hence, the differential equation of the streamlines is given by





The equation of continuity for incompressible fluid is given by



But equation (2) expresses the condition that the differential equation (1) should be exact.

Thus we can say that is a complete differential say , so that



The equation (3) gives

and 

The function  is called the stream function or the current function.

From equations (1) and (3), it is obvious that stream lines are obtained by integrating the equation



which on integrating gives,

.

This shows that the stream function is constant along stream line. It is therefore to be noted that the current function always exists in all types of two dimensional motion whether rotational or irrotational.

**Relation between stream function and velocity potential:** In the case of two dimensional irrotational motion of an incompressible inviscid fluid, the velocity potential  always exists such that

 and 

In the case of two dimensional rotational and irrotational motion, the stream function  always exists such that

 and 

From (1) and (2), we get

 and .

This are the required relations between stream function and velocity potential.

**Question-02:** Discuss the physical meaning of .

OR

Prove that the difference between the stream function at two points will express the flow crossing the line joining those points.

**Answer:** Consider a curve LM lying in the -plane and assume that  and  be the stream functions at L and M respectively. Let be the element of the curve at P and  the angle which the tangent at P makes with OX. Now, if and  are the velocities parallel to axes of  and , then we have velocities along inward normal





























.

The flow across the curve LM from right to left











 (**Proved**).

**Irrotational motion in two dimensions:** In case of two dimensional irrotational motion, the velocity potential  always exists such that

 and 

Hence the equation of continuity is



From (1) and (2), we get



This shows that  satisfies Laplace’s equation.

Further from the relation of stream function and velocity potential, we have

 and 

and 

But in general,  which gives,



which shows that  satisfies Laplace’s equation.

Also .

This shows that the families of curves given by  and  cut orthogonally.

**NOTE:** In the case of two dimensional motion of an incompressible inviscid fluid,

1. The stream function  exists whether the motion is rotational or irrotational.
2. The velocity potential  can exists only where the motion is irrotational.
3. In case of irrotational motion, and  satisfy Laplace’s equation and Cauchy Riemann equations.

**Question-03:** Define complex velocity potential.

**Answer: Complex velocity potential:** In the case of two dimensional irrotational motion of an incompressible inviscid fluid, the following relations always hold

 and 

where  and  are velocity potential and stream function respectively.

Also if , where  be analytic in a region R, then the relations given by equation (1) also hold, where  and  are called conjugate functions.

Here the function , in the case of two dimensional irrotational motion of an incompressible inviscid fluid, is called complex velocity potential.

**Complex Velocity:** If and , then





And 



From (1) and (2), we get









which is called complex velocity.

The equation (3) gives,



which gives the speed of the fluid particle at any point in the field of flow.

**Stagnation Point:** The points where the complex velocity is zero i.e.  are called stagnation points.

Thus at stagnation points, .

**Question-04:** Determine the velocity in polar coordinates for two dimensional and irrotational motion.

**Answer: Velocity in polar coordinates:** Let the Cartesian coordinates of a point P in an incompressible fluid be , where the motion is assumed to be two dimensional, irrotational. If  be the polar coordinates of the point P, then we have,

 and .

 ,  ,  , 













Also we know that,

 and 

Now 

 [by (1)]









and 













Thus we have,  and 

**Source:** If the motion of a fluid consists of symmetrical outward radial flow in all directions proceeding from a point, the point is known as a simple source. A source is thus a point at which fluid is continously created and distributed.

**Example**: Airbubble.

**Sink:** If the motion of a fluid consists of symmetrical inward radial flow to a point from all directions, the point is known as a simple sink. A sink is thus a point at which fluid is continously absorbed and annihilated. Thus a sink is a negative source.

**Example**: Whirlpool.

**Strength of source and sink:** If the total flow across a small curve surrounding the source is , then is called the strength of the source. Since sink is opposite of source so  is the strength of the sink.

**Doublet:** A combination of a source of strength  and a sink of strength  at a small distance  apart such that is finite is called a doublet.

If , a finite quantity, where is taken infinitely great and infinitely small then  is called the strength of the doublet; the axis of the doublet being in the sense from  to 

**Question-05:** Find complex potential of a source.

OR

For a source prove that the complex potential is 

**Answer:** Consider a source of strength at the origin O. So the total flow across a small curve surrounding the source is . Let be the radial velocity at distance from the source. Then the flow across a circle of radius is .

By the conservation of mass, we can write























Also we have 

From (1) and (2), we get





Integrating, 

Again, 



Integrating, 

Then the complex velocity potential is given by













This is the complex velocity potential due to a source at the origin.

If the source of strength  at , then the complex potential of this source is



Similarly, for source of strengths ,  at points the complex velocity potential is



.

**Question-06:** Derive the complex potential for a doublet.

OR

For a doublet prove that the complex potential is .

**Answer:** Let A and B denote the positions of the sink and source. Also let P be a point such that ,  , , .























Draw a perpendicular line  from  on  such that .

Then .



Now the velocity potential at P due to this combination is







 [Expanding  and neglecting higher order terms]









where  is the strength of the doublet.

We know, 







Again, 







Equations (2) and (3) shows that , which is possible only when is either zero or constant.

Thus 

Now the complex potential is













This is the complex velocity potential due to a doublet.

Further if the doublet makes an angle  with axis, then we have to write  for and the complex potential will be







If the doublet is at the point  then



Similarly at points  we have



.

**Problem**

**Problem-01:** A velocity field is given by . Find the stream function and the streamline for this field at .

**Solution:** Given that, .

Here 



From (1), we get





Integrating, 



Again, from (2), we get







Integrating, 

So equation (3) becomes,



This is the required stream function.

**2nd part:** The streamline for  is given by





which are rectangular hyperbola.

**Problem-02:** Find the stream function  for the given velocity field .

**Solution:** Given that, .

Here 



From (1), we get





Integrating, 



Again, from (2), we get





Integrating, 

So equation (3) becomes,



This is the required stream function.

**Problem-03:** A two dimensional flow field is given by . Then

1. Show that the flow is irrotational,
2. Find the velocity potential,
3. Verify that  and  satisfy Laplace equation
4. Find the streamlines and potential lines.

**Solution:** Given that, 

1. The velocity components are,

 and 

The velocity field is



Now .

Hence the motion is irrotational. (**Showed**)

1. We know that, 

and 

From (1), we get



Integrating, 

Differentiating (3) w.r.to , we get







Integrating, 

So equation (3) becomes,



This is the required velocity potential.

1. 

and .

Hence and  satisfy Laplace equation.

1. The streamlines,  and the potential lines,  , are given by

 and 

where  and  are constants.

**Problem-04:** Show that the velocity vector is everywhere tangent to lines in the plane along which .

**Solution:** The stream function  in plane can be expressed as



For streamlines, 





But from the definition of stream function, we get

 and 

Using (2) and (3), we get





which shows that the velocity vector is tangent to the lines .

**Problem-05:** Prove that when the speed is everywhere the same, the streamlines are straight.

**Solution:** The equation of the streamlines are given by,





Integrating, 

This implies that the intersection of these planes are straight lines. (**Proved**)

**Problem-06:** Find the streamlines, lines of equipotential, velocity of the complex potential

.

**Solution:** The complex potential is given by







Equating the real and imaginary parts, we get



and 

which are velocity potential and stream function respectively.

The streamlines are given by







Now  when or .

So  and  are streamlines and these give the motion of liquid in the angle between two perpendicular walls.

The lines of equipotential are given by







The velocity at any point is

.

**Problem-07:** Find the streamlines, lines of equipotential, velocity of the complex potential

.

**Solution:** The complex potential is given by





Put, ,  where  and .

The equation (1) becomes,







Equating the real and imaginary parts, we get



and 

which are velocity potential and stream function respectively.

The streamlines are given by







when  ,then







which give the motion of liquid in the region given by  .

The lines of equipotential are given by







The velocity at any point is

.

**Problem-08:** Determine the condition for which ,  will be the velocity components of an incompressible fluid. Show that for this motion the streamlines will be conic section in general and rectangular hyperbolas when the motion is irrotational.

**Solution:** Given that, and 

The necessary condition for possible fluid motion is





This is the required condition for the given functions to be velocity components of an incompressible fluid.

**2nd part:** we know, 

and 

From (2), we get



Integrating, 

Differentiating (3) w. r. to , we get



From (1) and (4), we get





Integrating, 

So equation (3) becomes,





For streamlines, 



.

This represents a conic (**Showed**).

**3rd part:** When the motion is irrotational, then







Hence the streamline equation in the irrotational flow is



 where 

which is clearly the equation of a rectangular hyperbola (**Showed**).

**Problem-09:** What arrangement of sources and sinks will give rise to the function ? Draw a rough sketch of the streamlines in this case and prove that two of them subdivide into the circle and the axis of .

**Solution:** Given that, 









which shows that there are two sinks of unit strength at  ,  and a source of unit strength at origin .

The above equation can be written as







Separating the real and imaginary part, we get the velocity potential



and the stream function











The streamlines are given by







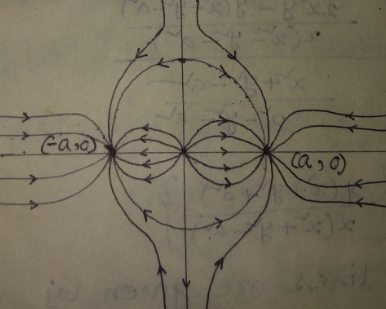
when the constant tends to infinity, the corresponding streamlines are given by





Thus the streamlines subdivide into a circle  i.e.  and the axis of  namely . Also if we take constant to be zero, we get  i.e. axis is a streamline.

Hence the axes and the circle are the streamlines. The other streamlines are sketched in the figure.



**Problem-10:** Two sources of each strength “m” are placed at the points and  and a sink of strength 2m is at the origin. Show that the streamlines are curves ; where  is parameter.

**Solution:** According to the question, we can construct the following figure,





















The complex potential is







; 

Separating real and imaginary parts, we get



and 

The streamlines are given by























 [Putting ]





 where 

 (**Showed**).

**Problem-11:** If there are sources at and  and sinks at and , all of equal strength m, then show that the streamlines through these four points is a circle.

**Solution:** According to the question, we can construct the following figure,



















The complex potential is













; 

Separating real and imaginary parts, we get



and 

The streamlines are given by



 [ Putting ]









when , we get





Thus the streamline is a circle (**Showed**).

**Problem-12:** A source and a sink of equal strength are placed at the points within a fixed circular cylinder . Show that the streamlines are given by ; where  is constant.

**Solution:** According to the question, we can construct the following figure,











In the absence of circular boundary, the complex potential due to a source of strength “m” (say) at  and a sink of strength “-m” at  is given by



when the circular boundary is inserted into the flow field, then the complex potential due to source and sink inside the boundary is given by



[By circle theorem, ]













Separating real and imaginary parts, we get

















The streamlines are given by











(**Showed**).

**Problem-13:** There is a source of strength m at the origin and equal sinks at the points  and . Discuss two dimensional motion due to the source and sinks.

**Solution:** According to the question, we can construct the following figure













Let “m” be the strength of the source at the origin O and “- m” be the strength of sinks at  and  respectively.

Now the complex potential at any point due to this system of source and sinks is







; 

This is the required complex potential.

Separating real and imaginary parts, we get



and 

These are the required velocity potential and stream function respectively.

The lines of equipotential are given by







.

The streamlines are given by













The fluid speed at any point in the flow field is













.

**Problem-14:** Find the stream function of two dimensional motion due to two equal sources and an equal sink midway between them. Sketch the streamlines and find the velocity at any point of the flow field.

**Solution:** According to the question, we can construct the following figure













Let “m” be the strength of sources at  , and “-m” be the strength of sink at the middle point of  respectively.

Now the complex potential at any point due to this system of sources and sink is







; 

Separating real and imaginary parts, we get



This is the required stream function.

The streamlines are given by





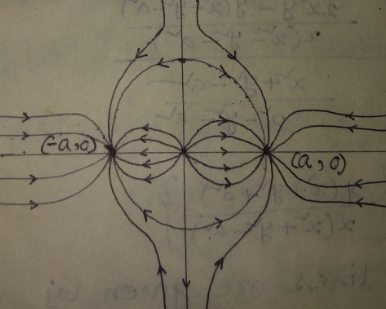








when then  is the streamline and when then  and  are the streamlines. The other streamlines are sketched in the figure.



The fluid speed at any point in the flow field is













.

**Problem-15:** Find Complex potential, Velocity potential, Stream function, Lines of equipotential and the fluid speed at  from the figure.





















**Solution:** According to the given figure, the complex potential is







; 

This is the required complex potential.

Separating real and imaginary parts, we get



and 

These are the required velocity potential and stream function respectively.

The lines of equipotential are given by







.

The fluid speed at any point in the flow field is













.

**Exercise**

**Problem-01:** A velocity field is given by . Find the stream function and the streamline for this field at .

**Problem-02:** Find the stream function  for the given velocity field .

**Problem-03:** A two dimensional flow field is given by . Show that the flow is irrotational and find the velocity potential. Also find the equation of the streamlines and potential lines.

**Problem-04:** If there are sources at and  and sinks at and all of equal strength m, then show that the streamlines through these four points is a circle.

**Problem-05:** Two sources of each strength “m” are placed at the points and  and a sink of strength 2m is at the origin. Show that the streamlines are curves ; where  is parameter.

**Problem-06:** A source and a sink of equal strength are placed at the points inside the circular cylinder . Show that the streamlines are given by ; where  is constant.